On the Total Graph of a Ring and Its Related Graphs: A Survey

Ayman Badawi

Abstract Let R be a (commutative) ring with nonzero identity and $Z(R)$ be the set of all zero divisors of R. The *total graph* of R is the simple undirected graph $T(\Gamma(R))$ with vertices all elements of R, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. This type of graphs has been studied by many authors. In this paper, we state many of the main results on the total graph of a ring and its related graphs.

Keywords Total graph • Zero divisors • Diameter • Girth • Connected graph Genus • Generalized total graph • Dominating set • Clique • Chromatic number

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1 Introduction

Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between ring-theoretic and graph-theoretic properties; see the recent survey articles [13, 32]. For example, as in [10], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$; see the articles [6, 11–12, 15–17, 19, 36]. The total graph (as in [7]) has been investigated in [2–5, 25, 32, 33, 35, 37]; and several

A. Badawi (✉)
Department of Mathematics and Statistics, American University of Sharjah, P.O. Box 26666 Sharjah, United Arab Emirates
e-mail: abadawi@aus.edu

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variants of the total graph have been studied in [1, 8, 9, 14, 16, 18, 21–24, 26, 27, 31]. The goal of this survey article is to enclose many of the main results on the total graph of a commutative ring and its related graphs.

Let G be a (simple) graph. We say that G is *connected* if there is a path between any two distinct vertices of G. At the other extreme, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). The eccentricity of a vertex x in G is the distance between x and the vertex which is at the greatest distance from x, $e(x) = \max\{d(x, y)|y$ is a vertex in GA̧$\}$. The radius of the graph G, $r(G)$, is defined by $r(G) = \min\{e(x)|x$ is a vertex in $G\}$, and the center of the graph is the set of all of its vertices whose eccentricity is minimal, i.e., it is equal to the radius. So, the radius of the graph is equal to the smallest eccentricity and diameter to the largest eccentricity of a vertex in this graph. It is well known that for connected graphs of diameter d and radius r, one has $r \leq d \leq 2r$. Recall that a *clique* in a graph is a set of pairwise adjacent vertices. The *clique number* of a graph G, denoted by $\omega(G)$, is the order of a largest clique in G. Also, $\chi(G)$ denotes the chromatic number of G and is the minimum number of colors which is needed for a proper coloring of G, i.e., a coloring of the vertices of G such that adjacent vertices have distinct colors. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,n}$ a *star graph*. We say that two (induced) subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (resp., G_2) is adjacent (in G) to any vertex not in G_1 (resp., G_2). By abuse of notation, we will sometimes write $G_1 \subseteq G_2$ when G_1 is a subgraph of G_2. A general reference for graph theory is [20].

Throughout this paper, all rings R are with $1 \neq 0$. Let R be a commutative ring with nonzero identity. Then $Z(R)$ denotes its set of zero divisors, $\mathrm{Nil}(R)$ denotes its ideal of nilpotent elements, $\mathrm{Reg}(R)$ denotes its set of nonzero divisors (i.e., $\mathrm{Reg}(R) = R \setminus Z(R)$), and $U(R)$ denotes its group of units. For $A \subseteq R$, let $A^* = A \setminus \{0\}$. We say that R is *reduced* if $\mathrm{Nil}(R) = \{0\}$, and $\dim(R)$ will always mean Krull dimension. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and $\mathbb{F}_q$ will denote the integers, rational numbers, integers modulo n, and the finite field with q elements, respectively. General references for ring theory are [29, 30].

2 The Total Graph of a Ring

In [7], Anderson and I defined the *total graph* of R to be the (undirected) graph $T(\Gamma(R))$ with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. Let $\mathrm{Reg}(T((\Gamma(R)))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $\mathrm{Reg}(R)$.

Theorem 2.1 ([7, Theorem 2.2]). *Let R be a commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$.*

1. *If $2 \in Z(R)$, then $\mathrm{Reg}(T(\Gamma(R))$ is the union of $\beta - 1$ disjoint K^{α}'s.*
2. *If $2 \notin Z(R)$, then $\mathrm{Reg}(T(\Gamma(R))$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Theorem 2.2 ([7, Theorem 2.4]). *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then*

1. $\mathrm{Reg}(T(\Gamma(R))$ *is complete if and only if either $R/Z(R) \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*
2. $\mathrm{Reg}(T(\Gamma(R))$ *is connected if and only if either $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$.*
3. $\mathrm{Reg}(T(\Gamma(R))$ *is totally disconnected if and only if R is an integral domain with* $\mathrm{char}(R) = 2$.

Theorem 2.3 ([7, Theorem 2.9]). *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then the following statements are equivalent:*

1. $\mathrm{Reg}(T(\Gamma(R))$ *is connected.*
2. *Either $x + y \in Z(R)$ or $x - y \in Z(R)$ for all $x, y \in \mathrm{Reg}(R)$.*
3. *Either $x + y \in Z(R)$ or $x + 2y \in Z(R)$ for all $x, y \in \mathrm{Reg}(R)$. In particular, either $2x \in Z(R)$ or $3x \in Z(R)$ (but not both) for all $x \in \mathrm{Reg}(R)$.*
4. *Either $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$.*

Theorem 2.4 ([7, Theorems 3.3, 3.4]). *Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then $T(\Gamma(R))$ is connected if and only if $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in Z(R)$. Furthermore, suppose that $T(\Gamma(R))$ is connected and let n be the least integer $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in Z(R)$. Then* $\mathrm{diam}(T(\Gamma(R))) = n$. *In particular, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then* $\mathrm{diam}(T(\Gamma(R))) = 2$.

In the following example, for each integer $n \geq 2$, we construct a commutative ring R_n such that $Z(R_n)$ is not an ideal of R_n and $T(\Gamma(R_n))$ is connected with $\mathrm{diam}(T(\Gamma(R))) = n$.

Example 2.5. Let $n \geq 2$ be an integer, $D = Z[X_1, X_2, \ldots, X_{n-1}]$, K be the quotient field of D, $P_0 = (X_1 + X_2 + Ą\cdots + X_{n-1})$, $P_i = (X_i)$ for each integer i with $1 \leq i \leq n-2$, and $P_{n-1} = (X_{n-1} + 1)$. Then $P_0, P_1, \ldots, P_{n-1}$ are distinct prime ideals of D. Let $F = P_0 \cup P_1 Ą\cup \cdots \cup P_{n-1}$; then $S = D\ F$ is a multiplicative subset of D. Set $R_n = D(+)(K/D_S)$. Then $Z(R_n) = F(+)(K/D_S))$. Since $(1, 0) = (-X_1 - X2 - \cdots - X_{n-1}, 0) + (X_1, 0) + (X_2, 0) + (X_3, 0) + Ą\cdots + (X_{n-1} + 1, 0)$ is the sum of n zero divisors of R_n, by construction we conclude that n is the least integer $m \geq 2$ such that 1 is the sum of m zero divisors of R_n. Hence $T(\Gamma(R_n)$ is connected with $\mathrm{diam}(T(\Gamma(R_n))) = n$ by Theorems 2.4 above.

Theorem 2.6 ([7, Theorem 3.1]). *If* $\mathrm{Reg}(\Gamma(R))$ *is connected, then $T(\Gamma(R))$ is connected.*

The converse of Theorem 2.6 is not true. We have the following example.

Example 2.7. Let $R = \mathbb{Q}[X](+)(\mathbb{Q}(X)/\mathbb{Q}[X])$. Then one can easily show that $Z(R) = (\mathbb{Q}[X]\ \mathbb{Q}^*)(+)(\mathbb{Q}(X)/\mathbb{Q}[X])$ is not an ideal of R and $\mathrm{Reg}(R) = U(R) = \mathbb{Q}^*(+)(\mathbb{Q}(X)/\mathbb{Q}[X])$. Thus $T(\Gamma(R))$ is connected with $\mathrm{diam}(T(\Gamma(R)) = 2$ (by Theorems 2.4) since $(1,0) = (X,0)(+)(X+1,0)$ with $(X,0), (X+1,0) \in Z(R)$. However, $\mathrm{Reg}(\Gamma(R))$ is not connected since there is no path from $(1,0)$ to $(2,0)$ in $\mathrm{Reg}(\Gamma(R))$.

Theorem 2.8. *1. [7, Corollary 3.5] If $T(\Gamma(R))$ is connected, then* $\mathrm{diam}(T(\Gamma(R)) = d(0,1)$.

2. [7, Corollary 3.5] If $T(\Gamma(R))$ is connected and $\mathrm{diam}(T(\Gamma(R)) = n$, *then* $\mathrm{diam}(\mathrm{Reg}(\Gamma(R))) \geq n-2$.

3. [4, Corollary 1] If R is a commutative Noetherian ring and $T(\Gamma(R))$ is connected with diameter n, then $n-2 \leq$ $\mathrm{diam}(\mathrm{Reg}(\Gamma(R))) \leq n$.

Theorem 2.9 ([8, Theorem 4.4]). *Let R be a commutative ring.*

(1) If R is either an integral domain or isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, then $\mathrm{gr}(T(\Gamma(R))) = \infty$.
(2) If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{gr}(T(\Gamma(R))) = 4$.
(3) Otherwise, $\mathrm{gr}(T(\Gamma(R))) = 3$.

Theorem 2.10 ([35, Theorem 2.1]). *Let R be a finite commutative ring with 1 such that $Z(R)$ is not an ideal of R. Then $r(T(\Gamma(R))) = 2$.*

Theorem 2.11 ([35, Theorem 2.2]). *Let R be a commutative ring with 1 such that $Z(R)$ is not an ideal of R, and let n be the smallest integer such that $1 = z_1 + \cdots + z_n$, for some $z_1, \ldots, z_n \in$ Ą$Z(R)$. Then $r(T(\Gamma(R))) = n$.*

Theorem 2.12 ([35, Theorem 3.2]). *Let R be a ring such that $Z(R)$ is not an ideal of R. Then $T(\Gamma(R[x]))$ is connected if and only if $T(\Gamma(R))$ is connected. Furthermore if* $\mathrm{diam}(T(\Gamma(R))) = n$, *then* $\mathrm{diam}(T(\Gamma(R[x]))) = r(T(\Gamma(R[x])) = n$.

Theorem 2.13 ([35, Theorem 3.4]). *Let R be a reduced ring such that $Z(R)$ is not an ideal of R. Then $T(\Gamma(R[[x]]))$ is connected if and only if $T(\Gamma(R))$ is connected. Furthermore if* $\mathrm{diam}(T(\Gamma(R))) = n$, *then* $\mathrm{diam}(T(\Gamma(R[[x]]))) = r(T(\Gamma(R[[x]])) = n$.

Let G be a simple undirected graph. Recall that a *Hamiltonian path* of G is a path in G that visits each vertex of G exactly once. A *Hamilton cycle (circuit)* of G is a Hamilton path that is a cycle. A graph G is called a *Hamilton graph* if it has a Hamilton cycle.

Theorem 2.14 ([4, Theorem 3]). *Let R be a finite commutative ring such that $Z(R)$ is not an ideal. Then the following statements hold:*

1. $T(\Gamma(R))$ is a Hamiltonian graph.
2. $\mathrm{Reg}(\Gamma(R))$ *is a Hamiltonian graph if and only if R is isomorphic to none of the rings: $\mathbb{Z}_2^{n+1}, \mathbb{Z}_2^n \times \mathbb{Z}_3, \mathbb{Z}_2^n \times \mathbb{Z}_4, \mathbb{Z}_2^n \times \mathbb{Z}_2[X]/(X^2)$, where n is a natural number.*

Theorem 2.15 ([25, Theorem 5.2]). *If R is a commutative ring and* diam $(T(\Gamma(R))) = 2$*, then $T(\Gamma(R))$ is Hamilton graph.*

Theorem 2.16 ([25, Corollary 5.3]). *If R is an Artinian ring, then $T(\Gamma(R))$ is Hamilton graph.*

Recall that a simple undirected graph is called a *planar graph* if it can be drawn on the plane in such way that no edges cross each other. Recall that a commutative ring R is called a *local* (*quasilocal*) *ring* if it has exactly one maximal ideal.

Theorem 2.17 ([33, Theorem 1.5]). *Let R be a finite commutative ring such that $T(\Gamma(R))$ is planar. Then the following statements hold:*

1. *If R is a local ring, then R is a field or R is isomorphic to one of the following rings:*
   $\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_2[X, Y]/(X, Y)^2, \mathbb{Z}_4[X]/(2X, X^2)$
   $\mathbb{Z}_4[X]/(2X, X^2 - 2), \mathcal{Z}_8, \mathbb{F}_4[X](X^2), \mathbb{Z}_4[X]/(X^2 + X + 1)$*, where $\mathbb{F}_4$ is a field with exactly four elements.*
2. *If R is not a local ring, then R isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$.*

A simple undirected nonplanar graph G is called *toroidal* if the vertices of G can be placed on a torus such that no edges cross.

Theorem 2.18 ([33, Theorem 1.6]). *Let R be a finite commutative ring such that $T(\Gamma(R))$ is toroidal. Then the following statements hold:*

1. *If R is a local ring, then R is isomorphic to either $\mathbb{Z}_9$ or $\mathbb{Z}_3/(x^2)$.*
2. *If R is not a local ring, then R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{F}_4$ is a field with exactly four elements.*

Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, k is an oriented surface with k handles. The genus of a graph G, denoted $G(G)$, is the minimal integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. Note that a graph G is a planar iff $g(G) = 0$ and G is toroidal iff $g(G) = 1$. Note that if x is a real number, then $\lceil x \rceil$ is the least integer that is greater than or equal to x.

Theorem 2.19 ([24, Theorem 3.2]). *Let R be a finite commutative ring with identity, I be an ideal contained in $Z(R)$, $|I| = n$ and $|R/I| = m$. Then the following statements are true:*

1. *If $2 \in I$, then $g(T(\Gamma(R))) \geq m\lceil \frac{(n-3)(n-4)}{12} \rceil$.*
2. *If $2 \notin I$, then $g(T(\Gamma(R))) \geq \lceil \frac{(n-3)(n-4)}{12} \rceil + (\frac{m-1}{2})\lceil \frac{(n-2)^2}{4} \rceil$.*

Theorem 2.20 ([24, Corollary 3.4]). *Let R be a finite commutative ring with identity such that $Z(R)$ is an ideal of R, $|Z(R)| = n$ and $|R/Z(R)| = m$. Then the following statements hold:*

1. *If* $2 \in Z(R)$, *then* $g(T(\Gamma(R))) = m\lceil \frac{(n-3)(n-4)}{12} \rceil$.
2. *If* $2 \notin I$, *then* $g(T(\Gamma(R))) = \lceil \frac{(n-3)(n-4)}{12} \rceil + (\frac{m-1}{2})\lceil \frac{(n-2)^2}{4} \rceil$.

Theorem 2.21 ([24, Theorem 4.3]). *Let* R *be a finite commutative ring. Then* $g(T(\Gamma(R))) = 2$ *if and only if* R *is isomorphic to either* $\mathbb{Z}_{10}$ *or* $\mathbb{Z}_3 \times \mathbb{F}_4$, *where* $\mathbb{F}_4$ *is a field with four elements.*

Let v be a vertex of a simple undirected graph G. Then the degree of v is denoted by $\deg(v)$. We say $\deg(v) = k$ if there are exactly k (distinct) vertices in G where each vertex is connected to v by an edge. Let G be a simple undirected graph. We say that G is *Eulerian* if it is connected and its vertex degrees are all even.

Theorem 2.22. 1. *[37, Theorem 3.3] Let* R *be a finite commutative ring. Then* $T(\Gamma(R))$ *is Eulerian if and only if* R *is isomorphic to a direct sum of two or more finite fields of even orders, i.e.,* $R \cong \bigoplus_{i=1}^{k} \mathbb{F}_{2^{t_i}}$ *for some* $k \geq 2$.
2. *[25, Lemma 5.1] Suppose that* $Z(R)$ *is not an ideal of* R. *Then* $T(\Gamma(R)$ *is Eulerian if and only if* $2 \in Z(R)$ *and* $|Z(R)|$ *is an odd integer.*

Let G be a simple undirected graph with V as its set of vertices. A subset S of V is called a *dominating set* of G if for every $a \in V \setminus S$, there is a $b \in S$ such that $a - b$ is an edge of the graph G. The domination number $\gamma(G)$ is the minimum size of a dominating set of G.

Theorem 2.23 ([37, Theorem 4.1]). *Let* R *be a finite commutative ring and* $n = \min\{|R/M| \mid M$ *is a maximal ideal of* $R\}$. *Then* $\gamma(T(\Gamma(R))) = n$, *except when* R *is a (finite) field of an odd order, where* $\gamma(T(\Gamma(R))) = \frac{n-1}{2} + 1$.

Let $H = \{d \mid d$ is a dominating set of $T(\Gamma(R))\}$. The *intersection graph of dominating sets* denoted by $IT(R)$ is a simple undirected graph with vertex set H and two distinct vertices a and b in H are adjacent if an only if $a \cap b = \emptyset$ (see [26, 27]).

Theorem 2.24 ([26, Theorem 3.1]). *Let* R *be a commutative Artinian ring with* $|R| \geq 4$ *and let* I *be an annihilator ideal of* R *such that* $|R/I|$ *is finite. Then*

1. $IT(R)$ *is connected and* $\mathrm{diam}(IT(R)) \leq 2$.
2. $\mathrm{gr}(IT(R))) \in \{3, 4\}$. *In particular,* $\mathrm{gr}(IT(R)) = 4$ *if and only if either* $R \cong \mathbb{Z}_4$ *or* $R \cong \mathbb{Z}_2[X]/(X^2)$.

Theorem 2.25 ([26, Theorem 3.2]). *Let* R *be a commutative Artinian ring with* $|R| \geq 4$ *and let* I *be an annihilator ideal of* R *such that* $|R/I|$ *is finite. Then*

1. $IT(R)$ *is a regular graph (i.e., all vertices in* $IT(R)$ *have the same degree).*
2. $IT(R)$ *is a complete graph if and only if* R *is an integral domain.*
3. $IT(R)$ *is a bipartite graph if and only if either* $R \cong \mathbb{Z}_4$ *or* $R \cong \mathbb{Z}_2[X]/(X^2)$.
4. $IT(R)$ *is a cycle if and only if either* $R \cong \mathbb{Z}_4$ *or* $R \cong \mathbb{Z}_2[X]/(X^2)$.

Theorem 2.26 ([26, Theorem 5.4]). *Let* R *be a finite commutative ring. Then*

1. $IT(R)$ *is planar if and only if* R *is isomorphic to either* $\mathbb{Z}_3$ *or* $\mathbb{Z}_4$ *or* $\mathbb{Z}_5$ *or* $\mathbb{Z}_2[x]/(X^2)$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{F}_{2^n}$ *(a field with* 2^n *elements) for some positive integer* $n \geq 1$).

2. $IT(R))$ *is toroidal if and only if* $R \cong \mathbb{Z}_6$.
3. $g(IT(R)) = 2$ *if and only if* $R \cong \mathbb{Z}_7$.

Theorem 2.27 ([26, Theorem 5.5]). *If R is a finite commutative ring, then* $g(IT(R)) \leq g(T(\Gamma(R)))$.

Theorem 2.28 ([27, Theorem 2.1]). *Let R be a commutative Artinian ring with $|R| \geq 4$ and assume that I is the unique annihilator ideal of R such that $|R/I|$ is minimum. Then $IT(R)$ is Eulerian if and only if R is not a field.*

Theorem 2.29 ([27, Theorem 2.2]). *Let R be a commutative Artinian ring with $|R| \geq 4$ and assume that I is an annihilator ideal of R such that $|R/I|$ is minimum. Then $IT(R)$ is a Hamilton graph.*

We recall that a graph G with number of vertices equals $m \geq 3$ is called *pancyclic* if G contains cycles of all lengths from 3 to m. Also G is called *vertex-pancyclic* if each vertex v of G belongs to every cycle of length l for $3 \leq l \leq m$.

Theorem 2.30. *Let R be a commutative Artinian ring with $|R| \geq 4$ and assume that I is an annihilator ideal of R such that $|R/I|$ is minimum. Then*

1. *[27, Theorem 2.3] $IT(R)$ is pancyclic if and only if either $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$.*
2. *[27, Corollary 2.1] $IT(R)$ is vertex-pancyclic if and only if neither $R \cong \mathbb{Z}_4$ nor $R \cong \mathbb{Z}_2[X]/(X^2)$ (i.e., $IT(R)$ is not pancyclic).*

We recall that a *perfect graph* is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Theorem 2.31. *Let R be a finite commutative ring. Then:*

1. *[27, Theorem 4.1]* $\chi(IT(R) = \omega(IT(R))$.
2. *[27, Theorem 4.2] $IT(R)$ is perfect if and only if either R is an integral domain or R has a unique annihilator ideal I with $|R/I| = 2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Let $CT(\Gamma(R))$ denotes the complement of the total graph of a commutative ring R, i.e., $CT(\Gamma(R))$ is a simple undirected graph with R as its vertex set, and two distinct vertices x, y in $CT(\Gamma(R))$ are adjacent if $x + y \in \mathrm{Reg}(R)$.

Recall that a *path graph* is a particularly simple example of a tree, namely a tree with two or more vertices that is not branched at all, that is, contains only vertices of degree 2 and 1. In particular, it has two terminal vertices (vertices that have degree 1), while all others (if any) have degree 2.

Theorem 2.32 ([25, Theorem 2.16]). *Let R be a commutative ring. Then the following statements are true:*

1. *$CT(\Gamma(R))$ is a path if and only if $R \cong \mathbb{Z}_2$.*
2. *$CT(\Gamma(R))$ is complete if and only if R is an integral domain and* $\mathrm{char}(R) = 2$.
3. *$CT(\Gamma(R)))$ is a star if and only if either $R \cong \mathbb{Z}_2$ or $R\mathbb{Z}_3$.*

4. *$CT(\Gamma(R))$ is a cycle if and only if either $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$ or $R \cong \mathbb{Z}_6$.*
5. *$CT(\Gamma(R))$ is a complete bipartite graph if and only if either R is a local ring [with maximal ideal $Z(R)$] such that $R/Z(R) \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*

Theorem 2.33. *Let R be a finite commutative ring. Then*

1. *[25, Corollary 4.5]* $\mathrm{gr}(CT(\Gamma(R)) = 3, 4, 6, \infty$.
2. *[25, Lemma 5.1] Suppose $Z(R)$ is not an ideal of R. Then $CT(\Gamma(R)$ is Eulerian if and only if $2 \in Z(R)$ and $|\mathrm{Reg}(R)|$ is an even integer.*

Let $C(R)$ represent a simple undirected graph with vertex set R and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x - y \in Z(R)$. It is natural for one to ask when is $T(\Gamma(R))$ isomorphic to $C(R)$? We have the following result.

Theorem 2.34 ([37, Theorem 5.2]). *Let R be a finite commutative ring. Then the two graphs $T(\Gamma(R))$ and $C(R)$ are isomorphic if and only if at least one of the following conditions is true:*

1. *$R \cong R_1 \oplus \cdots \oplus R_k, k \geq 1$, and each R_i is a local ring of an even order.*
2. *$R \cong R_1 \oplus \cdots \oplus R_k, k \geq 2$, and each R_i is a local ring such that $\min\{|R_i/M_i|$ where M_i is the maximal ideal of $R_i\} = 2$.*

Let R be a noncommutative ring. Then one can define $T(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ in the same way as for the commutative case. Let R be a ring. Then $M_n(R), GL_n(R)$, and $T_n(R)$ denote the set of $n \times n$ matrices over R, the set of $n \times n$ invertible matrices over R, and the set of $n \times n$ upper triangular matrices over R, respectively.

Theorem 2.35 ([35, Theorem 3.7]). *Let R be a commutative ring. The total graph $T(\Gamma(M_n(R)))$ is connected and* $\mathrm{diam}(T(\Gamma(M_n(R))) = 2$.

Theorem 2.36 ([3, Theorem 1]). *Let F be a field with $\mathrm{char}(F) \neq 2$ and n be a positive integer. Then $\omega(\mathrm{Reg}(\Gamma(M_n(F)))) < \infty$, and moreover* $\omega(\mathrm{Reg}(\Gamma(M_n(F)))) \leq \sum_{k=0}^{n} \frac{(n!)^2}{k![(n-k)!]^2}$.

Theorem 2.37 ([3, Theorem 2]). *For every field F with $\mathrm{char}(F) \neq 2$,* $\omega(\mathrm{Reg}(\Gamma(M_2(F)))) = 5$.

Theorem 2.38 ([3, Theorem 3]). *For every division ring D, $\mathrm{char}(D) \neq 2$,* $\mathrm{diag}(\pm 1, \ldots, \pm 1\} \ldots, \pm 1)$ (*the set of all diagonal matrices with diagonal entries in the set $\{-1, 1\}$ forms a maximal clique for* $\mathrm{Reg}(\Gamma(M_n(D)))$).

Theorem 2.39 ([5, Theorem 1]). *If F is a field, $\mathrm{char}(F) \neq 2$ and n is a positive integer, then $\chi(\mathrm{Reg}(\Gamma(T_n(F)))) = \omega(\mathrm{Reg}(\Gamma(T_n(F)))) = 2^n$.*

Theorem 2.40 ([2, Theorem 1, Theorem 3]). *Let R be a ring (not necessarily commutative). Then* $\mathrm{gr}(\mathrm{Reg}(\Gamma(R))), \mathrm{gr}(T(\Gamma(R))) \in \{3, 4, \infty\}$.

Recall that a *tree* is an undirected graph in which any two vertices are connected by exactly one simple path. In other words, any connected graph without simple cycles is a tree. A *forest* is a disjoint union of trees.

Theorem 2.41 ([2, Theorem 2]). *Let R be a left Artinian ring and $\mathrm{Reg}(\Gamma(R))$ be a tree. Then R is isomorphic to one of the following rings: $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2^r$, $\mathbb{Z}_3 \times \mathbb{Z}_2^r, \mathbb{Z}_4 \times \mathbb{Z}_2^r, \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2^r, T_2(\mathbb{Z}_2), T_2(\mathbb{Z}_2) \times \mathbb{Z}_2^r$, where $T_2(\mathbb{Z}_2)$ denotes the ring of 2×2 upper triangular matrices over $\mathbb{Z}_2$ and r is a natural number.*

Theorem 2.42 ([2, Theorem 5]). *Let R be a finite ring (not necessarily commutative). Then $\mathrm{Reg}(\Gamma(R))$ is regular (i.e., all vertices have the same degree).*

Theorem 2.43. *Let R be ring (not necessarily commutative). Then*

1. *[2, Theorem 7] If R is a left Artinian ring and $\mathrm{Reg}(\Gamma(R))$ contains a vertex adjacent to all other vertices, then $\mathrm{Reg}(\Gamma(R))$ is complete.*
2. *[2, Theorem 8] If $2 \notin Z(R)$ and $\mathrm{Reg}(\Gamma(R))$ is a complete graph, then $J(R) = 0$ (where $J(R)$ is the Jacobson radical of R).*
3. *[2, Theorem 9] If R is a left Artinian ring and $2 \notin Z(R)$, then $\mathrm{Reg}(\Gamma(R))$ is a complete graph, if and only if $R \cong \mathbb{Z}_3^r$, for some natural number r.*
4. *[2, Corollary 4] If R is a reduced left Noetherian ring and $2 \notin Z(R)$ such that $\mathrm{Reg}(\Gamma(R))$ is a complete graph, then $R \cong \mathbb{Z}_3^r$, for some natural number r.*

3 The Total Graph of a Commutative Ring Without the Zero Element

In this section, we consider the (induced) subgraph $T_0(\Gamma(R))$ of $T(\Gamma(R))$ obtained by deleting 0 as a vertex. Specifically, $T_0(\Gamma(R)))$ has vertices $R^* = R \setminus \{0\})$, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$.

Let $\mathrm{d}_T(x, y)$ (resp., $\mathrm{d}_{T_0}(x, y)$) denote the distance from x to y in $T(\Gamma(R))$ (resp., $T_0(\Gamma(R))$).

Theorem 3.1 ([8, Theorem 4.3]). *Let R be a commutative ring. Then* $\mathrm{diam}(T_0(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$.

Theorem 3.2 ([8, Theorem 4.5]). *Let R be a commutative ring.*

(1) If R is either an integral domain or isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{gr}(T_0(\Gamma(R))) = \infty$.
(2) If R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, then $\mathrm{gr}(T_0(\Gamma(R))) = 4$.
(3) Otherwise, $\mathrm{gr}(T_0(\Gamma(R))) = 3$.

Let $x, y \in R^*$ be distinct. We say that $x - a_1 - \cdots - a_n - y$ is a *zero-divisor path* from x to y if $a_1, \ldots, a_n \in Z(R)^*$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \leq i \leq n$ (let $x = a_0$ and $y = a_{n+1}$). We define $\mathrm{d}_Z(x, y)$ to be the length of a shortest zero-divisor path from x to y ($\mathrm{d}_Z(x, x) = 0$ and $\mathrm{d}_Z(x, y) = \infty$ if there is no such path) and $\mathrm{diam}_Z(R) = \sup\{\, d_Z(x, y) \mid x, y \in R^* \,\}$. In particular, if $x, y \in R^*$ are distinct and $x + y \in Z(R)$, then $x - y$ is a zero-divisor path from x to y with $\mathrm{d}(x, y) = 1$.

Let $\mathrm{Min}(R)$ denote the set of all minimal prime ideals of a commutative ring R. Recall that $U(R)$ denotes the set of all units of a commutative ring R.

Theorem 3.3 ([8, Theorem 5.1]). *Let R be a commutative ring that is not an integral domain. Then there is a zero-divisor path from x to y for every $x, y \in R^*$ if and only if one of the following two statements holds.*

(1) R is reduced, $|\mathrm{Min}(R)| \geq 3$, and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.
(2) R is not reduced and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

Moreover, if there is a zero-divisor path from x to y for every $x, y \in R^*$, then $\mathrm{diam}_Z(R) \in \{2, 3\}$ *and R is not quasilocal.*

Theorem 3.4 ([8, Theorem 5.2]). *Let R be a commutative ring. Then* $\mathrm{diam}_Z(R) \in \{0, 1, 2, 3, \infty\}$.

Theorem 3.5 ([8, Theorem 5.3]). *Let $R = R_1 \times R_2$ for commutative local (quasilocal) rings R_1, R_2 with maximal ideals M_1, M_2, respectively, and* $\mathrm{Nil}(R_2) \neq \{0\}$. *If there are $a_1 \in U(R_1)$ and $a_2 \in U(R_2)$ such that $(2a_1, 2a_2) \in U(R)$ and $(a_1, a_2) + (2a_1, 2a_2) \notin Z(R)$, then* $\mathrm{diam}_Z(R) = 3$.

Let $x, y \in R^*$ be distinct. We say that $x - a_1 - \cdots - a_n - y$ is a *regular path* from x to y if $a_1, \ldots, a_n \in \mathrm{Reg}(R)$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \leq i \leq n$ (let $x = a_0$ and $y = a_{n+1}$). We define $d_{\mathrm{reg}}(x, y)$ to be the length of a shortest regular path from x to y ($\mathrm{d}_{\mathrm{reg}}(x, x) = 0$ and $\mathrm{d}_{\mathrm{reg}}(x, y) = \infty$ if there is no such path), and $\mathrm{diam}_{\mathrm{reg}}(R) = \sup\{\, d_{\mathrm{reg}}(x, y) \mid x, y \in R^* \,\}$. In particular, if $x, y \in R^*$ are distinct and $x + y \in Z(R)$, then $x - y$ is a regular path from x to y with $\mathrm{d}_{\mathrm{reg}}(x, y) = 1$. Note that $\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_2) = 0$, $\mathrm{diam}_{\mathrm{reg}}(\mathbb{Z}_3) = 1$, and $\mathrm{diam}_{\mathrm{reg}}(R) = \infty$ for any other integral domain R. We also have $\max\{\, \mathrm{diam}(T(\Gamma(R))), \mathrm{diam}(\mathrm{Reg}(\Gamma(R))) \,\} \leq \mathrm{diam}_{\mathrm{reg}}(R)$.

Theorem 3.6 ([8, Theorem 5.6]). *Let R be a commutative ring with* $\mathrm{diam}(T_0(\Gamma((R))) = n < \infty$.

(1) Let $u \in U(R)$, $s \in R^*$, and P be a shortest path from s to u of length $n - 1$ in $T_0(\Gamma(R))$. Then P is a regular path from s to u.
(2) Let $u \in U(R)$, $s \in R^*$, and $P : s - a_1 - \cdots - a_n = u$ be a shortest path from s to u of length n in $T_0(\Gamma(R))$. Then either P is a regular path from s to u, or $a_1 \in Z(R)^*$ and $a_1 - \cdots - a_n = u$ is a regular path of length $n - 1 = d_{T_0}(a_1, u)$.

Theorem 3.7 ([8, Theorem 5.7]). *Let R be a commutative ring.*

(1) If $s \in \mathrm{Reg}(R)$ and $w \in \mathrm{Nil}(R)^*$, then there is no regular path from s to w. In particular, if there is a regular path from x to y for every $x, y \in R^*$, then R is reduced.
(2) If R is reduced and quasilocal, then there is no regular path from any unit to any nonzero nonunit in R.

In particular, if there is a regular path from x to y for every $x, y \in R^*$, then R is reduced and not quasilocal.

Recall from [28] that a commutative ring R is a *p.p. ring* if every principal ideal of R is projective. For example, a commutative von Neumann regular ring is a p.p. ring, and $\mathbb{Z} \times \mathbb{Z}$ is a p.p. ring that is not von Neumann regular. It was shown in [34, Proposition 15] that a commutative ring R is a p.p. ring if and only if every element of R is the product of an idempotent element and a regular element of R (thus a commutative p.p. ring that is not an integral domain has nontrivial idempotents).

Theorem 3.8 ([8, Theorem 5.9, Corollary 5.10]). *Let R be a commutative p.p. ring that is not an integral domain. Then there is a regular path from x to y for every $x, y \in R^*$. Moreover,* $\mathrm{diam}_{\mathrm{reg}}(R) = 2$. *In particular, if R be a commutative von Neumann regular ring that is not a field, then there is a regular path from x to y for every $x, y \in R^*$ and* $\mathrm{diam}_{\mathrm{reg}}(R) = 2$.

Theorem 3.9 ([8, Theorem 5. 14]). *Let R be a commutative ring that is not an integral domain. Then there is a regular path from x to y for every $x, y \in R^*$ if and only if R is reduced,* $\mathrm{Reg}(\Gamma(R))$ *is connected, and for each $a \in Z(R)^*$ there is a $b \in Z(R)^*$ such that $d_z(a, b) > 1$ (it is possible that $d_z(a, b) = \infty$).*

Theorem 3.10 ([8, Corollary 5.15]). *Let R be a reduced commutative ring such that* $\mid \mathrm{Min}(R) \mid= 2$. *Then there is a regular path from x to y for every $x, y \in R^*$ if and only if* $\mathrm{Reg}(\Gamma(R))$ *is connected.*

4 Generalized Total Graph

A subset H of R becomes a *multiplicative-prime* subset of R if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$, and (ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For example, H is multiplicative-prime subset of R if H is a prime ideal of R, H is a union of prime ideals of R, $H = Z(R)$, or $H = R \setminus U(R)$. In fact, it is easily seen that H is a multiplicative-prime subset of R if and only if $R \setminus H$ is a saturated multiplicatively closed subset of R. Thus H is a multiplicative-prime subset of R if and only if H is a union of prime ideals of R [30, Theorem 2]. Note that if H is a multiplicative-prime subset of R, then $\mathrm{Nil}(R) \subseteq H \subseteq R \setminus U(R)$; and if H is also an ideal of R, then H is necessarily a prime ideal of R. In particular, if $R = Z(R) \cup U(R)$ (e.g., R is finite), then $\mathrm{Nil}(R) \subseteq H \subseteq Z(R)$.

Let H be a multiplicative-prime subset of a commutative ring R. the *generalized total graph* of R, denoted by $GT_H(R)$, as the (simple) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in H$. For $A \subseteq R$, let $GT_H(A)$ be the induced subgraph of $GT_H(R)$ with all elements of A as the vertices. For example, $GT_H(R \setminus H)$ is the induced subgraph of $GT_H(R)$ with vertices $R \setminus H$. When $H = Z(R)$, we have that $GT_H(R)$ is the so-called total graph of R as introduced in [7] and denoted there by $T(\Gamma(R))$. As to be expected, $GT_H(R)$ and $T(\Gamma(R))$ share many properties. However, the concept of generalized total graph, unlike the earlier concept of total graph, allows us to study graphs of integral domains.

Theorem 4.1 ([9, Theorem 4.1]). *Let H be a prime ideal of a commutative ring R, and let $|H| = \alpha$ and $|R/H| = \beta$.*

1. *If $2 \in H$, then $GT_H(R \setminus H)$ is the union of $\beta - 1$ disjoint K^{α}'s.*
2. *If $2 \notin H$, then $GT_H(R \setminus H)$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Theorem 4.2 ([9, Theorem 4.2]). *Let H be a prime ideal of a commutative ring R.*

1. *$GT_H(R \setminus H)$ is complete if and only if either $R/H \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*
2. *$GT_H(R \setminus H)$ is connected if and only if either $R/H \cong \mathbb{Z}_2$ or $R/H \cong \mathbb{Z}_3$.*
3. *$GT_H(R \setminus H)$ (and hence $GT_H(H)$ and $GT_H(R)$) is totally disconnected if and only if $H = \{0\}$ (thus R is an integral domain) and $\mathrm{char}(R) = 2$.*

The next theorem gives a more explicit description of the diameter and girth of $GT_H(R \setminus H)$ when H is a prime ideal of R.

Theorem 4.3 ([9, Theorem 4.4]). *Let H be a prime ideal of a commutative ring R.*

1. a. $\mathrm{diam}(GT_H(R \setminus H)) = 0$ *if and only if $R \cong \mathbb{Z}_2$.*
   b. $\mathrm{diam}(GT_H(R \setminus H)) = 1$ *if and only if either $R/H \cong \mathbb{Z}_2$ and $R \not\cong \mathbb{Z}_2$ (i.e., $R/H \cong \mathbb{Z}_2$ and $|H| \geq 2$), or $R \cong \mathbb{Z}_3$.*
   c. $\mathrm{diam}(GT_H(R \setminus H)) = 2$ *if and only if $R/H \cong \mathbb{Z}_3$ and $R \not\cong \mathbb{Z}_3$ (i.e., $R/H \cong \mathbb{Z}_3$ and $|H| \geq 2$).*
   d. *Otherwise,* $\mathrm{diam}(GT_H(R \setminus H)) = \infty$.
2. a. $\mathrm{gr}(GT_H(R \setminus H)) = 3$ *if and only if $2 \in H$ and $|H| \geq 3$.*
   b. $\mathrm{gr}(GT_H(R \setminus H)) = 4$ *if and only if $2 \notin H$ and $|H| \geq 2$.*
   c. *Otherwise,* $\mathrm{gr}(GT_H(R \setminus H)) = \infty$.
3. a. $\mathrm{gr}(GT_H(R)) = 3$ *if and only if $|H| \geq 3$.*
   b. $\mathrm{gr}(GT_H(R)) = 4$ *if and only if $2 \notin H$ and $|H| = 2$.*
   c. *Otherwise,* $\mathrm{gr}(GT_H(R)) = \infty$.

The following examples illustrate the previous theorem.

Example 4.4 ([9, Example 4.5]). (a) Let $R = \mathbb{Z}$ and H be a prime ideal of R. Then $GT_H(R \setminus H)$ is complete if and only if $H = 2\mathbb{Z}$, and $GT_H(R \setminus H)$ is connected if and only if either $H = 2\mathbb{Z}$ or $H = 3\mathbb{Z}$. Moreover, $\mathrm{diam}(GT_H(R \setminus H)) = 1$ if and only if $H = 2\mathbb{Z}$, and $\mathrm{diam}(GT_H(R \setminus H)) = 2$ if and only if $H = 3\mathbb{Z}$. Let $p \geq 5$ be a prime integer and $H = p\mathbb{Z}$. Then $GT_H(R \setminus H)$ is the union of $(p-1)/2$ disjoint $K^{\omega,\omega}$'s; so $\mathrm{diam}(GT_H(R \setminus H)) = \infty$. Finally, $\mathrm{diam}(GT_H(R \setminus H)) = \infty$ when $H = \{0\}$.
Also, $\mathrm{gr}(GT_H(R \setminus H)) = \infty$ if $H = \{0\}$, $\mathrm{gr}(GT_H(R \setminus H)) = 3$ if $H = 2\mathbb{Z}$, and $\mathrm{gr}(GT_H(R \setminus H)) = 4$ otherwise. Moreover, $\mathrm{gr}(GT_{\{0\}}(R)) = \infty$ and $\mathrm{gr}(GT_H(R)) = 3$ for any nonzero prime ideal H of R.
(b) Let $R = \mathbb{Z}_{pm} \times R_1 \times \cdots \times R_n$, where $m \geq 2$ is an integer, p is a positive prime integer, and $R_1, \ldots, R_n$ are commutative rings. Then $H = p\mathbb{Z}_{pm} \times R_1 \times \cdots \times R_n$

is a prime ideal of R. The graph $GT_H(R \setminus H)$ is complete if and only if $p = 2$, and $GT_H(R \setminus H)$ is connected if and only if $p = 2$ or $p = 3$. Moreover, $\mathrm{diam}(GT_H(R \setminus H)) = 1$ if and only if $p = 2$, and $\mathrm{diam}(GT_H(R \setminus H)) = 2$ if and only if $p = 3$. Assume that $p \geq 5$. Then $GT_H(R \setminus H)$ is the union of $(p-1)/2$ disjoint $K^{\alpha,\alpha}$'s, where $\alpha = m|R_1| \cdots |R_n|$; so $\mathrm{diam}(GT_H(R \setminus H)) = \infty$.

Also, $\mathrm{gr}(GT_H(R \setminus H)) = 3$ if $p = 2$ and $\mathrm{gr}(GT_H(R \setminus H)) = 4$ otherwise. Moreover, $\mathrm{gr}(GT_H(R)) = 3$ for any prime p.

Theorem 4.5 ([9, Theorem 4.7]). *Let H be a prime ideal of a commutative ring R. Then the following statements are equivalent.*

1. *$GT_H(R \setminus H)$ is connected.*
2. *Either $x + y \in H$ or $x - y \in H$ for every $x, y \in R \setminus H$.*
3. *Either $x + y \in H$ or $x + 2y \in H$ for every $x, y \in R \setminus H$. In particular, either $2x \in H$ or $3x \in H$ (but not both) for every $x \in R \setminus H$.*
4. *Either $R/H \cong \mathbb{Z}_2$ or $R/H \cong \mathbb{Z}_3$.*

Theorem 4.6 ([9, Theorem 5.1(3)]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R. If $GT_H(R \setminus H)$ is connected, then $GT_H(R)$ is connected.*

Theorem 4.7 ([9, Theorem 5.2, Theorem 5.3]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R. Then $GT_H(R)$ is connected if and only if $1 = z_1 + \cdots + z_n$, for some $z_1, \ldots, z_n \in H$. In particular, if H is not an ideal of R and either $\dim(R) = 0$ (e.g., R is finite) or R is an integral domain with $\mathrm{diam}(R) = 1$, then $GT_H(R)$ is connected. Furthermore, suppose that $G_H(R)$ is connected. Let $n \geq 2$ be the least integer such that $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in H$. Then $\mathrm{diam}(GT_H(R)) = n$. In particular, if H is not an ideal of R and either $\dim(R) = 0$ (e.g., R is finite) or R is an integral domain with $\dim(R) = 1$, then $\mathrm{diam}(GT_H(R)) = 2$.*

Theorem 4.8 ([9, Corollary 5.5]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R such that $GT_H(R)$ is connected.*

1. *$\mathrm{diam}(GT_H(R)) = d(0, 1)$.*
2. *If $\mathrm{diam}(GT_H(R)) = n$, then $\mathrm{diam}(GT_H(R \setminus H)) \geq n - 2$.*

Theorem 4.9 ([9, Theorem 5.15)]). *Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R.*

1. *Either $\mathrm{gr}(GT_H(H)) = 3$ or $\mathrm{gr}(GT_H(H)) = \infty$. Moreover, if $\mathrm{gr}(GT_H(H)) = \infty$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = Z(R)$; so $GT_H(H)$ is a $K^{1,2}$ star graph with center 0.*
2. *$\mathrm{gr}(GT_H(R)) = 3$ if and only if $\mathrm{gr}(GT_H(H)) = 3$.*
3. *$\mathrm{gr}(GT_H(R)) = 4$ if and only if $\mathrm{gr}(GT_H(H)) = \infty$ (if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$).*
4. *If $\mathrm{char}(R) = 2$, then $\mathrm{gr}(GT_H(R \setminus H)) = 3$ or ∞. In particular, $\mathrm{gr}(GT_H(R \setminus H)) = 3$ if $\mathrm{char}(R) = 2$ and $GT_H(R \setminus H)$ contains a cycle.*

5. $\mathrm{gr}(GT_H(R \setminus H)) = 3, 4,$ *or* ∞. *In particular,* $\mathrm{gr}(GT_H(R \setminus H)) \leq 4$ *if* $GT_H(R \setminus H)$ *contains a cycle.*

Let R be a commutative ring. Recall that a subset S of R is called a *multiplicatively closed subset* of R if S is closed under multiplication. A multiplicatively closed subset S of R is called *saturated* if $xy \in S$ implies that $x \in S$ and $y \in S$.

Let S be multiplicatively closed subset of a commutative ring R. The graph $\Gamma_S(R)$ is a simple undirected graph with all elements of R as vertices, and two distinct vertices x and y of R are adjacent if and only if $x + y \in S$.

Theorem 4.10 ([18, Corollary 1.6]). *Suppose that S is an ideal of R with $|S| = n$ and $|R/S| = m$.*

1. *If $2 \in S$, then $\Gamma_S(R)$ is the union of m disjoint K^n's.*
2. *If $2x \notin S$ for each $x \in R$, then $\Gamma_S(R)$ is the union of K^n with $(m-1)/2$ disjoint $K^{n,n}$'s.*

Theorem 4.11 ([18, Proposition 2.1]). *The graph $\Gamma_S(R)$ is complete if and only if $S = R$ or* $(\mathrm{char} R = 2$ *and* $S = R \setminus \{0\})$.

Theorem 4.12 ([18, Proposition 2.1]). *Let S be a saturated multiplicatively closed subset of R with $R\ S = \cup_{i=1}^{n} P_i$ such that $|R/P_i| = 2$ for some i. Then $\Gamma_S(R)$ is a bipartite graph. Furthermore, $\Gamma_S(R)$ is a complete bipartite graph if and only if $n = 1$.*

Theorem 4.13 ([18, Theorem 2.15]). *Let R be finite commutative ring and S be a saturated multiplicatively closed subset of R. Then* $\mathrm{gr}(\Gamma_S(R)) \in \{3, 4, 6, \text{Ą}\infty\}$.

The following is an example of saturated multiplicatively closed sets, to show that each of the numbers $3, 4, 6$, and ∞ given in the previous theorem can appear as the girth of some graphs.

Example 4.14 ([18, Example 2.16]). Let $R = \mathbb{Z}_6$. Then $\mathrm{gr}(\Gamma_{Z(R)}(R)) = 3$, $\mathrm{gr}(\Gamma_{U(R)}(R)) = 6$, and $\mathrm{gr}(\Gamma_S(R)) = 4$, where $S = \{1, 3, 5\}$. For the saturated multiplicatively closed subset $S = \{-1, 1\}$ of $\mathbb{Z}$, we have $\mathrm{gr}(\Gamma_S(R)) = \infty$.

Theorem 4.15 ([18, Theorem 2.17]). *Let R be finite and S be a saturated multiplicatively closed subset of R. Then* $\mathrm{gr}(\Gamma_S(R)) = \text{Ą}\infty$ *if and only if one of the following statements holds:*

1. $R = \mathbb{Z}_3$.
2. $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ *and* $|S| = 1$.

Theorem 4.16 ([18, Theorem 2.23]). *Let R be a finite commutative ring. For a saturated multiplicatively closed subset S of R, we have* $\mathrm{diam}(\Gamma_S(R)) \in \{1, 2, 3, \text{Ą}\infty\}$.

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